PSY 503: Foundations of Psychological Methods

Lecture 12: Estimation and Uncertainty II

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Central Limit Theorem

The distribution of the sample mean approaches the normal distribution as the sample size increases.

Central Limit Theorem

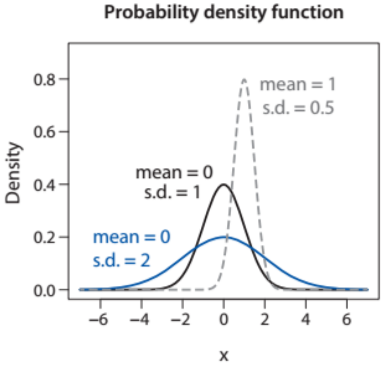
- Kicks in regardless of the distribution of the random variable
- This result is incredibly useful because the normal distribution is a parametric distribution
- This allows us to quantify the uncertainty of our estimates!

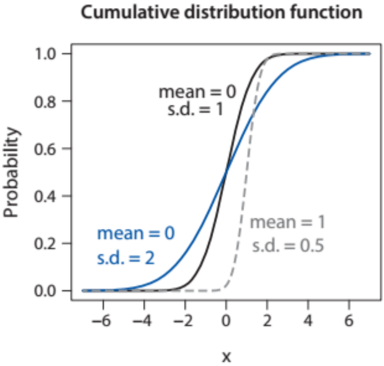
The normal distribution

- Also called the *Gaussian distribution*
- Can take any number on the real line $(-\infty, \infty)$
  - Continuous distribution
- Two parameters:
  - mean μ
  - standard deviation σ
- If X is a random variable, we may write

$$X \sim N(\mu, \sigma^2)$$

The normal distribution: PDF and CDF





- PDF: bell shaped, centered and symmetric around μ
- Standard deviation "controls" for the spread of the distribution
- Different means shift the PDF and CDF without changing their shape
- Larger standard deviations mean more variability

FYI: PDF of the normal distribution

$$f(x;(\mu,\sigma)) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2\Big), \quad -\infty \leq x \leq \infty$$

Standard normal distribution

The *standard normal distribution* is a normal distribution with $\mu = 0$ and $\sigma = 1$

The normal distribution: Properties

1. Adding a constant to (substracting a constant from) a normal random variable yields another normal random variable
2. Multiplying / dividing a random variable by a constant yields another normal random variable

The normal distribution: Properties

Let $X \sim N(\mu, \sigma^2)$. Let c be a constant. Then the following properties hold:

1. A random variable defined by $Z = X + c$ also follows a normal distribution, with $Z \sim N(\mu + c, \sigma^2)$
2. A random variable defined by $Z = cX$ also follows a normal distribution, with $Z \sim N(c\mu, (c\sigma)^2)$

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As a result,

$$\text{z-score} = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

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Then the **central limit theorem** states that the sample mean converges in distribution to the normal distribution. We write

$$\bar{X_n} \to N(\mathbb{E}[X], \frac{\mathbb{V}[X]}{n})$$

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- Confidence level often written $(1 - \alpha) * 100\%$, where α can take any value between 0 and 1
  - e.g., $\alpha = .05$ corresponds to the 95% confidence level

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- 95 % confidence Interval is

$$[\bar{X}_n - 1.96 \times \text{standard error}, \bar{X}_n + 1.96 \times \text{standard error}]$$

Confidence intervals

Let's open R Studio. . .

Confidence intervals and p-values

- Two sides of the same coin
- Suppose a random sample, a sample mean, and a null hypothesis that the sample mean is different from k. We will reject the null hypothesis at the significance level α (i.e., $p < \alpha$) if and only if the confidence interval with confidence level $1 - \alpha$ does not include k
- This applies to difference-in-means estimator
  - e.g., We reject the null hypothesis that there is no difference between the treatment and control groups at the 95% confidence level if and only if the 95% CI of the difference-in-means does not include zero, which corresponds to a p-value for a hypothesis test lower than .05

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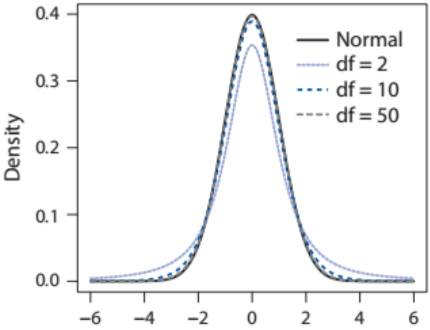

- Small samples: more conservative test
  - t-distribution has fatter tails
  - coverage is more conservative

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This implies that until your sample is large enough for the CLT to kick in, t-tests won't work unless you assume that X is normally distributed. Most often NOT the case!

Illustration: Difference-in-means estimator

Go to R Studio...