# PSY 503: Foundations of Psychological Methods
# Lecture 6: Basics of Probability Theory

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# What is Probability theory?

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Probability theory is the study of **random processes** (a.k.a., **random generative processes**, **random phenomena**).

## Random processes: Intuition

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THEN

- You are dealing with a random process

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Random processes are **mechanisms** that produce (probabilistic) outcomes... from **a world of possible** outcomes... with some degree of **uncertainty** but with **regularity**.

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  - The set of all possible outcomes
  - "world of probabilistic outcomes"
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- **Event:** $A$, $B$, $C$, etc.
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- **Probability:** Proportion of times an event or set of events will occur if you keep repeating the random process

# Examples of random processes

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- Random assignment of $N$ individuals to an experimental condition
- Random draw of a sample of $n$ individuals from a population of $N$ individuals
- Rolling a die

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- What if we assigned two individuals to Treatment (T) vs. Control (C)
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# Events

An *event* is a subset of the sample space $\Omega$ and corresponds to the realization of one or more than one outcomes $\omega$

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- Another example?

# Notations

| Syntax | Description |
|---|---|
| $\Omega$ | sample space |
| $\omega$ | a possible probabilistic outcome |
| $A \cup B$ | $A$ or $B$ |
| $A \cap B$ | $A$ and $B$ |
| $A^C$ | not $A$ |
| $A_1 \cup A_2 \cup \ldots \cup A_n$ | at least one of $A_1, \ldots, A_n$ |
| $A_1 \cap A_2 \cap \ldots \cap A_n$ | all of $A_1, \ldots, A_n$ |
| $A \cap B = \emptyset$ | $A$ and $B$ are mutually exclusive |

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- We randomly assign 8 participants to T vs. C
  - Possible outcome: $\omega = \mathrm{TTTTCCTC}$
  - Sample space: Set of all possible strings of length 8 of T's and C's

# Practice with events

- Let's **randomly** generate a possible outcome $\omega_j$ in R

```
sample(c("T", "C"),
       size = 8,
       replace = TRUE)
```

```
[1] "T" "T" "C" "C" "T" "T" "T" "C"
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- In the background, does R draw from this sample space?
- NO: Keep in mind that R draws an outcome $\omega_j$ from $\Omega = \{T, C\}$ 8 times in a row with replacement

# More practice with events

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- Let $C_1$ be the event that the first participant is assigned to the control condition
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- We could express this more generally:
  - $C_1 = \{(C,\ \omega_2,\ ...,\ \omega_8) : \omega_j \in \{\text{T}, \text{C}\} \text{ for } 2 \leq j \leq 8\}$

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- Let $A$ be the event, that at least one participant was assigned to control condition. We can write:

$$A = C_1 \cup C_2 \cup ... \cup C_8 \tag{1}$$

i.e., Participant 1 or participant 2 or . . . or participant 8 was assigned to control

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- We could have written:

$$A = \bigcup_{i=1}^{8} C_i \tag{2}$$

# More practice with events

$$B = \bigcap_{i=1}^{8} C_i \quad (3)$$

In plain English?

# Naive probability of an event

Let $A$ be an event with a finite sample space $\Omega$. The *naive probability* of $A$ is

$$P(A) = \frac{|A|}{|\Omega|} \tag{4}$$

in which |A| is the number of possible outcomes $\omega$ that satisfy A, and $|\Omega|$ is the total number of possible outcomes $\omega$ within $\Omega$.

# Wait, why is this naive?

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- Requires $\Omega$ to be finite
- Requires each possible outcome $\omega$ to have the same weight
  - This can be misleading!
  - e.g., polls, attrition

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# Probability model: Definition

- The **probability model** of a random phenomenon is the mathematical representation of this phenomenon. It includes:
  - All of the possible outcomes included in the sample space
  - The probability of each possible probabilistic outcome $\omega$ included in the sample space
- This is all that there is to know about a random phenomenon
- Very powerful: Contain enough information to predict with certainty the percentage of times that an outcome $\omega$ will happen if we repeat the random generative process many (many, many, many) times

# Probability model: Intuition

Overall, the probability of an outcome $\omega$ is the percentage of times that this outcome will happen if we repeat the random generative process:

- over and over again
- independently
- under the exact same conditions

# Probability model: Example

- The probability model of rolling a fair die includes:
  - Its sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$
  - The probability of each possible outcome $\omega_j$ is: $\mathrm{P}(w_j) = \frac{1}{6}$

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If we wonder what are the possible outcomes of rolling a fair die, we simply need to look at the probability model to realize that there are 6 possible outcomes.

If we wonder how likely it is that we will get a 6 after rolling a fair die, again, we can look at its probability model and learn that the probability of getting a 6 is $\frac{1}{6}$

# Probability Rules

# Probability rule # 1

- Probabilities take values between 0 and 1 (inclusive)
- For some event $A$:

$$0 \leq P(A) \leq 1$$

- Probability cannot be negative
- Probability cannot be greater than 1

# Probability rule # 2

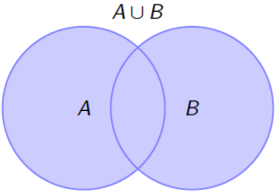
- Since $\Omega$ is the entire sample space,

$$P(\Omega) = 1$$

- e.g., What is the probability of getting an even or an odd number after rolling a fair die?

# Probability rule # 3

- The probability that $A$ **or** $B$ occurs is the probability of the **union** of $A$ and $B$
- Denoted by $P(A \cup B)$



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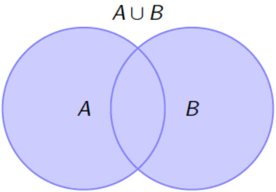
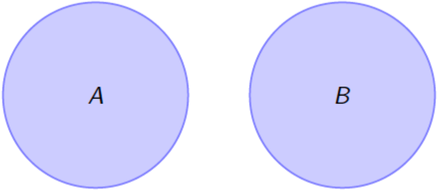



- Addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

# Mutually exclusive events

- Two events $A_i$ and $A_j$ are **mutually exclusive** (or **disjoint**) if they cannot happen at the same time



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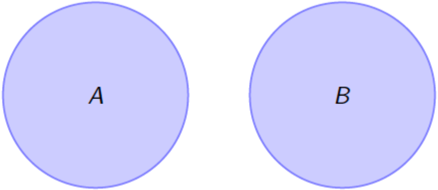
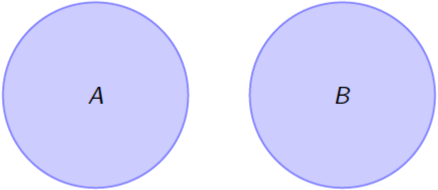



- For $i \neq j$, we have:

$$A_i \cap A_j = \emptyset$$

# Probability rule # 3 for mutually exclusive events



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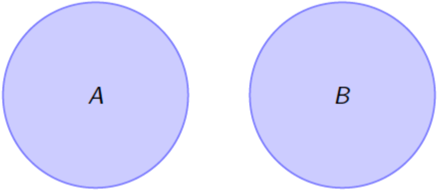


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- But what is $P(A \cap B)$?

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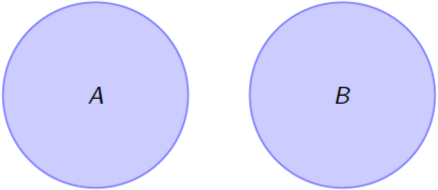


- Under the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- But what is $P(A \cap B)$? $P(A \cap B) = 0$
- Therefore,

$$P(A \cup B) = P(A) + P(B)$$

# Probability rule # 3 generalized to any number of mutually exclusive events

- Given any number of *mutually exclusive* events $A_1,\ A_2,\ ...,\ A_n$, the probability that one of these events will occur is the sum of their individual probabilities:
    - $P(A_1\ \cup\ A_2\ \cup\ ...\ \cup\ A_n) = P(A_1)\ + P(A_2)\ + ... +\ P(A_n)$

# Probability rule # 3 generalized to any number of mutually exclusive events

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  - $P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n)$
- Let $F$ be the event of rolling a fair die and getting an even number
  - $F = \{2, 4, 6\} = 2 \cup 4 \cup 6$
  - $P(F) = P(2) + P(4) + P(6)$

# Probability rule # 4

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- This implies

$$P(A^c) = 1 - P(A)$$