PSY 503: Foundations of Psychological Methods

Lecture 7: Conditional Probabilities and Independence

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If two events A and B are independent, knowing that A occurred does not inform the chances that B occurred. We have:

$$P(A|B) = P(A) \tag{1}$$

$$P(B|A) = P(B) \tag{2}$$

Conditional probability is everywhere

- Examples?

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- Causal effects
  - If a treatment d_i has an effect on Y_i, knowing that d_i occurred tells us something about the probability that Y_i occurs.
  - If a training improves performance at a test, knowing that an individual took that drug before the test provides information about the probability that this individual passed vs. failed.

Definition

If A and B are events with $P(B) > 0$, the *conditional probability* of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{3}$$

Illustration

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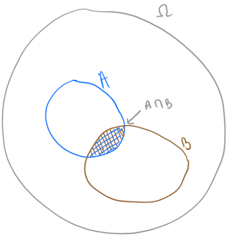


- Conditional probabilities allow us to update the probability of an event A based on the occurrence of another event
- $P(A)$ is called the *prior probability*
- $P(A|B)$ is called the *posterior probability*

Multiplication rule

$$P(A|B)P(B) = P(A \cap B) \tag{4}$$

The multiplication rule follows directly from Equation (3).

Implication

If A and B are independent events, we have:

$$P(A \cap B) = P(A)P(B) \tag{5}$$

Practice with grant proposal

You are about to send a grant proposal to an organization. While you read about the grant, you realize that your grant proposal will be sent to 5 different referees, who can be either social or cognitive psychologists. Imagine that for each grant proposal, the committee flips a coin five times and assigns the proposal to a social psychologist every time the flip returns heads, and to a cognitive psychologist every time the flip returns tails.

Assume an infinite pool of social and cognitive psychologists. What are the chances that your grant proposal is assigned to 5 cognitive psychologists?

Practice with grant proposal

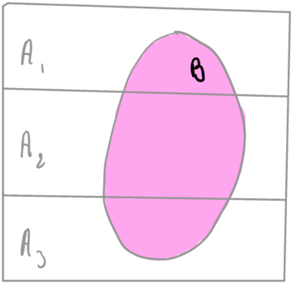
Let C_i be the event that your grant proposal is assigned to a cognitive psychologist. Since the events are independent from each other, we have:

$$\begin{aligned} P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5) &= P(C_1) \times P(C_2) \times P(C_3) \times P(C_4) \times P(C_5) \\ &= (\frac{1}{2})^5 \\ &= \frac{1}{32} \end{aligned}$$

Law of Total Probability (LTP)

Let $\{A_1, A_2, ..., A_n\}$ be a partition of Ω. This means that the A_i events are mutually exclusive and their union is Ω. If $P(A_i) > 0$ for all i, then

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) \tag{6}$$



LTP: Two events

In the special case of two events, we have:

$$P(B) = P(B \cap A) + P(B \cap A^C)$$

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$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{7}$$

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- Accuracy of test:
  - 85% of the people who have covid-19 test positive
  - 96% of the people who do not have covid-19 test negative
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  - $P(Pos|Covid) = .85$
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- Let:
  - $Covid$ be the event of having Covid-19
  - Pos be the event of testing positive
  - Neg be the event of testing negative
- We know that:
  - $P(Covid) = \frac{1}{130}$
  - $P(Pos|Covid) = .85$
  - $P(Neg|Covid^C) = .96$
- We need to calculate:
  - $P(Covid|Pos)$

Illustration: Covid-19

- Bayes' Rule:

$$P(Covid|Pos) = \frac{P(Pos|Covid)P(Covid)}{P(Pos)}$$

- $P(Pos)$?

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- Bayes' Rule:

$$P(Covid|Pos) = \frac{P(Pos|Covid)P(Covid)}{P(Pos)}$$

- $P(Pos)$?

- Use Law of Total Probability

Illustration: Covid-19

$$
\begin{aligned}
P(Pos) &= P(Pos \cap Covid) + P(Pos \cap Covid^C) \\
&= P(Pos|Covid)P(Covid) + P(Pos|Covid^C)P(Covid^C) \\
&= 0.85 \times \frac{1}{130} + (1 - 0.96) \times (1 - \frac{1}{130}) \\
&= 0.0462
\end{aligned}
$$

We can now derive $P(Covid|Pos)$ using Bayes' rule:

$$P(Covid|Pos) = \frac{.85 * \frac{1}{130}}{.0462} = .142$$

Lessons from Bayes' rule

- Based on the results of this test, the probability that your friend actually has covid-19 is .142
  - That's a 14.2% chance of actually having covid-19.
- Bayes' rule often yields counter-intuitive results!
- Importance of base rates

Probability Theory vs. Statistical Inference

Probability Theory

- We know the probability model of the random generative process
- We ask: given the data generative process, what data is likely?

Probability Theory

- For any given random phenomenon, probability theory is a set of tools that assume prior knowledge of:
  - The sample space
  - The probability of a set of events defined on that sample space
- Allows you to find the probability of any other possible event from that sample space

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- We usually don't know the probability model
- OK, we can find the probability of every outcome in the sample space by observing many many repetitions
- BUT most random phenomena cannot be repeated again, again, and again
- We generally need to infer the probability of each possible outcome using information on a few realizations of the random phenomenon of interest

Statisical Inference

- For any given random phenomenon, statistical inference is a set of tools that uses knowledge of:
  - A finite number of realizations of a random phenomenon
- ... to tell you how to make educated guesses about:
  - The sample space
  - The probability of events defined on that sample space

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  - Assuming that the probability of observing heads in a coin flip is 0.5
  - What is the probability of observing HHTT if we flip a coin four times?
- Statistical inference asks:
  - Suppose that you flip a coin four times and observe HHTT
  - What is your best guess for the probability of observing heads when flipping that coin? How confident are you in your guess?

Typical Probability Theory Question

- In the 2012 US presidential elections, 51.1% of voters voted in favor of Barack Obama
- If you were to randomly select 1,000 people that voted in this election, what is the probability that at least 500 of them voted in favor of Barack Obama?
- In other words, if you were to take many many random samples of 1000 people from the population of voters, in which fraction of these samples would you have at least 500 people that voted in favor of Barack Obama?

Typical Statistical Inference Question

- Before the ellection, FiveThirtyEight listed the results for multiple election polls:
- Using the information from Ipsos, what is your best guess on the total fraction of voters that would vote for Clinton if the elections were to happen now? How confident are you in your guess?
- Would your guess be different if you were to use information from the ABC News/Washington Post poll? What if you were to combine information from both?
- Example relevant to psychology research?

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- The typical probability theory and statistical inference questions are very hard to answer intuitively
- Kahneman and Tversky showed that humans tend to make a cognitive error that they called the *Law of Small Numbers*
- This cognitive error has a probability theory manifestation:
  - People underestimate the probability of sequential streaks occurring by chance
- and a statistical inference manifestation:
  - People overestimate the information about underlying probabilities contained in short sequences of outcomes from a random phenomenon. Even worse, they suggest theories to explain these regularities!

Law of Small Numbers: Illustration

In the 2010 NBA finals between the Boston Celtics and the LA Lakers, Celtic guard Ray Allen made seven three-point shots in a row. This is an example of what journalists call "hot streak". Observers usually tend to theorize about the reasons underlying hot streaks and cold streaks.

Law of Small Numbers

- Example of cognitive errors that justify the existence and the relevance of the theory of probability:
  - http://www.radiolab.org/story/91686-a-very-lucky-wind/ (listen to fragment from 10:00 to 15:00)